ON THE EVOLUTIONARY STATE OF β CEPHEI STARS

RICHARD STOTHERS
Institute for Space Studies, Goddard Space Flight Center, NASA
New York, New York

Received July 28, 1964; revised October 12, 1964

ABSTRACT

An attempt is made to explain the β Cephei variables as non-rotating stars undergoing radial oscillations, on the basis of their relatively low observed rotational velocities and the period analyses by van Hoof. Arguments based on the central condensation and on the time scale of evolution vis-à-vis their observed numbers indicate that these B0.5–B2 giants are in the hydrogen-burning phase. Model sequences are constructed for stars of 15 and 20 $M_\odot$ to the end of hydrogen burning. The pulsational characteristics are then obtained by perturbing the stable models in the usual adiabatic approximation. The results show that, at some central hydrogen abundance which is higher for the lower masses, the periods and their ratios may be accounted for. Ideally, these quantities imply a unique mass and mean molecular weight for each observed star on the H-R diagram, but comparison of theory and observation gives at present merely a mass range of 10–20 $M_\odot$ and probably a "normal" chemical composition. At any rate, the hypothesis of evolution of main-sequence O9–B1 stars *across* the instability strip seems to be correct. Uncertainties in the semiconvective theory of massive stars would appear to be irrelevant since the quasi-stable zone has almost no effect on the pulsational eigenfrequencies. Several lines of observational evidence tend to confirm the theoretical results that β Cephei stars of lower mass fall closer to the initial main sequence. The Wolf-Rayet objects form an apparent extension of the instability strip to higher masses (O stars).

I. INTRODUCTION

The β Cephei stars are a group of short-period intrinsic variables. They lie slightly above the main sequence at spectral types B0.5–B2 and exhibit slow or sometimes no rotation. The assumption that their variability may be explained by purely radial oscillations has come into difficulty accounting for the observed period-density relation (Ledoux and Walraven 1958; Gurm 1963). Moreover, since many of the more rapidly rotating members show a beat period, various attempts have been made, with some degree of success, to interpret them as stars undergoing non-radial oscillations (e.g., Ledoux 1951, 1958; Chandrasekhar and Lebovitz 1962; Böhm-Vitense 1963). Recently Porfir'ev has made a novel suggestion of "rotational" oscillations (variable meridional circulation) due to a combination of rotation and radial oscillations (Oliinik and Porfir'ev 1963).

In the interest of investigating fully the purely radial hypothesis, it seems worthwhile, still, to reconsider the variables as non-rotating stars in radial oscillation. Struve (1955*b*) has emphasized that many of the β Cephei stars are relatively simple variable stars, completely analogous to other variables that are believed to be in purely radial oscillation. In fact, van Hoof (1962*e*) has shown that many of the observed features of the β Cephei phenomenon may be explained by the interference of several simultaneously excited modes of radial oscillation.

It is the purpose of this paper to determine at which evolutionary stage massive stars become β Cephei variables, and to see whether the relevant observations may be explained on the assumption of radial pulsation. Thus we are here investigating the theoretical and observational consequences of assuming the correctness of van Hoof's analysis.

II. PRELIMINARY EVOLUTIONARY CONSIDERATIONS

Struve (1955*a*) originally suggested that stars with a certain critical mass evolve *up* the β Cephei strip. However, from a modern discussion by Schmalberger (1960), the β Cephei stars seem to lie on the H-R diagram near the locus of a range of masses at the end of hydrogen burning. This locus is defined by the first turn back toward the main sequence. Thus the β Cephei stars appear to have evolved from main sequence O9–B1 stars. Since calculated luminosities for models of upper main-sequence stars are fairly accurate, we may use the observed period–absolute-magnitude relation (see Sec. VII) to obtain the period-mass relation: Π (hours) $= 0.35\ M/M_\odot$, where the mass range is roughly 10–20 $M_\odot$. The major uncertainty lies in the bolometric corrections to the observed magnitudes. There are no observed masses of these stars.

If the β Cephei stars are indeed situated along the locus of secondary contraction (as suggested also by Kopylov 1959), then some ambiguity arises because the evolutionary track for models of massive stars swings back quickly to the right after the brief turn back, forming an S-shaped curve. The luminosity becomes only slightly higher, but the internal structure is drastically changed from a core-burning configuration to a contracting core with a surrounding, hydrogen-burning shell (Sakashita, Ôno, and Hayashi 1959; Hayashi and Cameron 1962; Stothers 1963, 1964, hereinafter called "Paper I" and "Paper II," respectively). The fully contracting phase was first suggested by Reddish in this connection (Reddish and Sweet 1960).

Two lines of evidence point strongly to the former configuration for the β Cephei stars. The first line of evidence is based on the observed statistics of these stars. If our sample of eighteen stars (van Hoof 1962*e*) is complete within at most a radius of 1 kpc around the Sun, in a Galaxy of effective $R = 10$ kpc, and if there is a maximum of 1×10^5 O–B2 stars in the Galaxy, then the number ratio of β Cephei stars to all O–B2 stars will be greater than $\frac{1}{50}$. In fact, McNamara and Hansen (1961) have obtained $\frac{1}{9}$. Now this ratio should be equal to the ratio of time spent in the β Cephei state to that spent in hydrogen burning. According to Hayashi and Cameron (1962), the gravitational (core) contraction phase of a star of 15.6 $M_\odot$ is $\frac{1}{200}$ as long as the hydrogen-burning phase; if hydrogen exhaustion is counted with the gravitational contraction phase, the ratio becomes $\frac{1}{80}$. Both ratios are far smaller than the observed number ratio.

The second line of evidence concerns the period ratios, according to the observational work of van Hoof and the results to be derived in this paper. During the hydrogen-burning phase, the calculated and observed period ratios agree at some characteristic central hydrogen abundance, dependent on the stellar mass. However, increasing central condensation leads to more and more discrepant values for these ratios.

Hence we conclude that we must seek models for the β Cephei stars in the hydrogen-burning phase of evolution.

III. STABLE MODELS

a) Basic Physics

The general structure assumed for models of massive stars has been outlined in Paper I. Here we adopt the same assumptions, notations, and equations as before. The adopted masses are 15 and 20 $M_\odot$, since at 10 $M_\odot$ electron scattering is no longer a good approximation to the opacity throughout the star. The initial composition is again taken to be

$$X_e = 0.70\,, \qquad Y_e = 0.27\,, \qquad Z_e = 0.03\,, \qquad X_{\rm CNO} = Z_e/2\,. \tag{1}$$

The parameters in the formula for nuclear-energy generation are, in the present case,

$$\begin{aligned} T_c &< 37 \times 10^6\,, \qquad \nu = 16\,, \qquad \log\ \epsilon_0 = -114.1\,, \\ T_c &> 37 \times 10^6\,, \qquad \nu = 15\,, \qquad \log\ \epsilon_0 = -106.6\,. \end{aligned} \tag{2}$$

b) Integration of Equilibrium Equations

The construction of models proceeds in the manner outlined in Paper I. However, the fitting was accomplished automatically by the computer in a procedure analogous to that described by Ezer and Cameron (1963). Integrations are performed from the surface inward through Zones I, III, and IV with trial values of the envelope eigenparameter (log C for the homogeneous model and λ for the inhomogeneous models), until $U < 3$ and $dU/dq < 0$ at some small prechosen mass fraction. An extrapolated value of β_c is then obtained, whereupon outward and inward integrations are begun, with the fitting accomplished in U and V at some β_f just inside the convective core. Finally the semiconvective Zone II is integrated and fitted to Zone III as shown in Paper I.

c) *Results for Stable Models*

Table 1 contains the essential results for evolutionary sequences of six models obtained for stars of 15 and 20 $M_\odot$. Comparison may be made with the analogous sequence calculated for 30 $M_\odot$ in Paper I. It should be noted that in the present paper the adopted values of $L_\odot$ and $R_\odot$ are those of Allen (1963), whereas in Papers I and II Chandrasekhar's (1939) values were used. In all comparisons with other work, we shall renormalize luminosities and radii to Allen's values whenever necessary.

Apart from the increasing importance of the semiconvective zone from 15 to 30 $M_\odot$, the only other point worthy of special mention is that the initial decrease of central density for stars of intermediate and high mass becomes negligible or even vanishes for very massive stars, at some mass between 20 and 30 $M_\odot$ (cf. also Henyey, LeLevier, and Levée 1959). It seems that the nuclear-energy generation is not quite sufficient to expand the central regions against their slow gravitational contraction. This would be due to the lower value of the temperature exponent, ν, which necessitates an increasing central density in order to help maintain the pressure gradient and sustain the high luminosity.

IV. PULSATING MODELS

a) *Pulsation Equation*

The equation describing small, radial adiabatic pulsations of a self-gravitating gaseous sphere may be written

$$\frac{\partial^2 \xi}{\partial r^2}+\frac{\partial \xi}{\partial r}\left[\frac{2}{r}+\frac{1}{\Gamma_1 P}\frac{\partial}{\partial r}(\Gamma_1 P)\right]+\xi\left[\frac{\sigma^2 r\rho+4g\rho}{\Gamma_1 P r}-\frac{2}{r^2}+\frac{2}{\Gamma_1 P r}\frac{\partial}{\partial r}(\Gamma_1 P)\right]=0\,, \tag{3}$$

where ξ is the radial displacement, $\sigma/2\pi = \Pi^{-1}$ the frequency, and Π the period of oscillation. We have also the adiabatic exponent

$$\Gamma_1=\beta+\frac{(4-3\beta)^2(\gamma-1)}{\beta+12(\gamma-1)(1-\beta)} \tag{4}$$

for a mixture of perfect gas and radiation (Chandrasekhar 1939). Equation (3) is the equation derived by Ledoux (1939) if the relative amplitude $\delta r/r$ is introduced in place of $\xi \equiv \delta r$. The boundary conditions are $\delta r = 0$ at $r = 0$ and $\delta P = 0$ at $r = R$.

It will be convenient to evaluate an explicit expression for $\partial(\Gamma_1 P)/\partial r$. Since the ionization zones of hydrogen and helium are of negligible extent in massive stars, we may take $\gamma = \frac{5}{3}$ throughout the whole star. Then with the help of the expression for Γ_1, the equation of state, and the definitions of the invariants V and n (Paper I), we obtain

$$\frac{1}{\Gamma_1 P}\frac{\partial}{\partial r}(\Gamma_1 P)=-\frac{V}{r}(1+b)\,, \tag{5}$$

where

$$b=\frac{(1-\beta)(n-3)}{B(n+1)}\,,\qquad B=\frac{\Gamma_1(8-7\beta)}{7\Gamma_1-8-2\beta}\,. \tag{6}$$

Table 1

Evolutionary Models during Hydrogen-Burning

	15 $M_\odot$						20 $M_\odot$					
Model	0	1	2	3	4	5	0	1	2	3	4	5
og C	-3.397	-3.300	-3.200	-3.100	-3.075	-3.067	-3.452	-3.400	-3.300	-3.200	-3.150	-3.140
	.....	0.822	0.969	1.098	1.185	1.21	.....	0.872	0.976	1.155	1.231	1.28
og q_1	.....	-0.331	-0.316	-0.299	-0.294	-0.293	.....	-0.282	-0.261	-0.239	-0.227	-0.225
og $q_{2,3}$	.....	-0.344	-0.347	-0.352	-0.354	-0.354	.....	-0.292	-0.296	-0.305	-0.311	-0.313
og q_4	-0.343	-0.414	-0.505	-0.622	-0.660	-0.673	-0.291	-0.327	-0.404	-0.499	-0.563	-0.578
2	.....	0.700	0.699	0.695	0.694	0.693	.....	0.700	0.698	0.690	0.683	0.681
3	.....	0.697	0.691	0.679	0.674	0.674	.....	0.699	0.688	0.666	0.650	0.645
4	0.700	0.537	0.331	0.093	0.025	0.004	0.700	0.611	0.419	0.193	0.062	0.035
c	0.887	0.859	0.819	0.760	0.740	0.733	0.843	0.824	0.781	0.719	0.678	0.669
og T_c	7.521	7.537	7.558	7.603	7.639	7.686	7.539	7.547	7.565	7.597	7.631	7.647
og ρ_c	0.737	0.730	0.752	0.861	0.966	1.110	0.622	0.620	0.629	0.693	0.788	0.836
og $(L/L_\odot)$	4.400	4.498	4.598	4.698	4.723	4.731	4.720	4.772	4.872	4.972	5.022	5.032
og $(R/R_\odot)$	0.630	0.693	0.780	0.899	0.923	0.897	0.710	0.742	0.819	0.928	1.001	1.013
og T_e	4.548	4.541	4.523	4.488	4.482	4.498	4.589	4.586	4.572	4.542	4.518	4.515
	0.700	0.632	0.560	0.494	0.479	0.474	0.700	0.656	0.573	0.493	0.455	0.447
$(10^6$ years)	0.00	3.55	6.52	8.68	9.12	9.25	0.00	1.52	3.96	5.83	6.58	6.72

Introducing equation (5) and $g = GM(r)/r^2$ into equation (3), we rewrite the pulsation equation as

$$\frac{\partial^2 \xi}{\partial r^2}+\frac{\partial \xi}{\partial r}\left[\frac{2}{r}-\frac{V}{r}(1+b)\right]+\xi\left[\frac{\sigma^2}{G}\frac{V r}{\Gamma_1 M(r)}-\frac{2}{r^2}-\frac{2V}{r^2}\left(1+b-\frac{2}{\Gamma_1}\right)\right]=0\,. \tag{7}$$

This is the final form of the pulsation equation, taking radiation pressure exactly into account. In the case of no radiation pressure ($1-\beta=0$) we have $b=0$ and $\Gamma_1=\gamma$, and equation (7) reduces to the equation used by Stothers and Schwarzschild (1961).

In terms of the non-dimensional envelope variables defined in Paper I equation (7) becomes

$$\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial \xi}{\partial x}\left[\frac{2}{x}-\frac{V}{x}(1+b)\right]+\xi\left[\omega^2\frac{V x}{\Gamma_1 q}-\frac{2}{x^2}-\frac{2V}{x^2}\left(1+b-\frac{2}{\Gamma_1}\right)\right]=0\,, \tag{8}$$

where

$$\omega^2=\sigma^2\frac{R^3}{GM} \tag{9}$$

is the eigenvalue of the problem. For a homologous sequence of stellar models ω^2 is constant, so that we obtain the familiar period–root-mean-density relation, $Q=\Pi\sqrt{(\bar{\rho}/\bar{\rho}_\odot)}=$ const.

In the core of the star, the pulsation equation may be written as equation (8) with the envelope variables simply replaced by the starred core variables of Paper I. In this case the eigenvalue is

$$\omega^{*2}=\sigma^2\frac{r_0{}^3}{GM_0}=\omega^2\frac{\beta_c\bar{\rho}}{3\rho_c}\,. \tag{10}$$

b) *Integration of Pulsation Equation*

The solution of the pulsation equation was expanded, as usual, in a power series around $r=0$ and $r=R$. For the numerical integrations a scheme similar to that used in fitting the stable models was adopted in order to obtain the correct value of ω^2. Integrations from the surface inward are performed for trial values of ω^2 until ξ and $\partial\xi/\partial x$ are suitably small near the center. Improvement is then obtained by fitting separate envelope and core integrations at β_f, with the help of $(1/\xi)(\partial\xi/\partial\beta)$ as a (single) fitting parameter since ω^2 and ω^{*2} are directly related by equation (10). It may be noted that this parameter is continuous at a density discontinuity (Interface II–III, Paper I).

c) *Results for Pulsating Models*

The pulsational characteristics of massive stars are collected in Table 2. For comparative purposes we have the analytical relation (Ledoux and Walraven 1958)

$$\omega_0{}^2\leq(3\bar{\Gamma}_1-4)\int_0^1\frac{q\,dq}{x}\bigg/\int_0^1 x^2 dq\,, \tag{11}$$

where the integrals represent the degree of central condensation. If the central condensation is not too large, the equality sign provides a good approximation for $\omega_0{}^2$ (Ledoux and Pekeris 1941). However, the higher modes will not be as sensitive to Γ_1 as $\omega_0{}^2$ is (Ledoux and Walraven 1958).

In confirmation of equation (11), Tables 1 and 2 show that $\omega_0{}^2$ and the period ratios decrease with β and hence Γ_1. This may be seen by comparing the initial models for the three masses, among which the differences in degree of central condensation, as indicated by $\rho_c/\bar{\rho}$, are surprisingly slight. The decrease is similar to that found for the standard model (Ledoux and Walraven 1958).

Even quantitatively the standard model (polytrope of index $n=3$) provides reasonable values for the pulsational characteristics, as evidenced by Table 3. The tabulated

Table 2

Pulsational Characteristics of Evolutionary Models for Massive Stars

	15 $M_{\odot}$						20 $M_{\odot}$						30 $M_{\odot}$					
Model	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5
$\log(R/R_{\odot})$	0.630	0.693	0.780	0.899	0.923	0.897	0.710	0.742	0.819	0.928	1.001	1.013	0.817	0.855	0.943	1.090	1.118	1.143
$\rho_c/\bar{\rho}$	20.0	30.5	58.2	171	257	299	20.0	24.8	43.2	107	221	266	20.3	27.2	52.5	178	239	316
X_c	0.700	0.537	0.331	0.093	0.025	0.004	0.700	0.611	0.419	0.193	0.062	0.035	0.700	0.583	0.373	0.112	0.069	0.030
ω_o^2	5.31	6.00	6.85	7.63	7.80	7.85	4.75	5.10	5.89	6.77	7.19	7.27	3.97	4.37	5.22	6.34	6.49	6.63
Π_o (hours)	2.75	3.21	4.06	5.79	6.24	5.67	3.31	3.57	4.33	5.90	7.37	7.64	4.28	4.65	5.76	8.70	9.47	10.2
Q_o (days)	0.050	0.047	0.044	0.042	0.042	0.041	0.053	0.051	0.048	0.045	0.043	0.043	0.058	0.056	0.051	0.046	0.045	0.045
Π_1/Π_o	0.612	0.650	0.686	0.709	0.713	0.713	0.588	0.610	0.654	0.690	0.701	0.703	0.549	0.580	0.633	0.681	0.686	0.690
Π_2/Π_o	0.452	0.484	0.515	0.539	0.543	0.544	0.432	0.450	0.486	0.519	0.531	0.533	0.402	0.426	0.469	0.511	0.515	0.519
Π_3/Π_o	0.360	0.386	0.413	0.433	0.437	0.438	0.344	0.359	0.389	0.416	0.426	0.428	0.319	0.338	0.374	0.409	0.413	0.416
Π_4/Π_o	0.300	0.322	0.346	0.363	0.366	0.367	0.287	0.299	0.324	0.348	0.357	0.358	0.266	0.281	0.312	0.342	0.345	0.348
Π_2/Π_1	0.739	0.744	0.751	0.760	0.761	0.762	0.736	0.738	0.744	0.752	0.756	0.757	0.732	0.734	0.740	0.750	0.751	0.753
Π_3/Π_1	0.589	0.594	0.602	0.611	0.613	0.614	0.586	0.588	0.594	0.603	0.608	0.609	0.581	0.583	0.590	0.601	0.602	0.603
Π_4/Π_1	0.491	0.496	0.503	0.512	0.513	0.514	0.488	0.490	0.496	0.504	0.508	0.509	0.484	0.485	0.492	0.502	0.503	0.504

Table 3

Interpolated Model Characteristics for the Phase of Equal Calculated and Observed Period Ratios

	$15\ M_\odot$	$20\ M_\odot$	$30\ M_\odot$	Standard ($\Gamma_1 = 1.5$
$\log(L/L_\odot)$	4.57	4.94	5.40	....
$\log T_e$	4.53	4.55	4.56	....
q_4	0.33	0.34	0.35	....
τ (10^6 years)	5.6	5.2	4.6	....
$\log(R/R_\odot)$	0.75	0.89	1.10	....
$\rho_c/\bar{\rho}$	45	70	200	54.2
X_c	0.39	0.27	0.1	....
ω_o^2	6.6	6.5	6.4	7.04
Π_o (hours)	3.7	5.2	9	....
Q_o (days)	0.045	0.046	0.046	0.044
Π_1/Π_o	0.675	0.678	0.68	0.687

values are derived from exact calculations for $\Gamma_1 = 1.54$ (Schwarzschild 1941). The period ratios of the first four modes of the standard model for various values of Γ_1 agree almost exactly with corresponding ratios for the models in Table 2; the eigenvalues ω_0^2 show reasonable agreement. Thus pulsationally, for a suitable choice of Γ_1 the standard model is a good approximation to the exact models.

Although the effective polytropic index, n, departs significantly from 3 below the stellar surface in the case of the exact models, it is little affected by the increasing central condensation, as comparison of Figure 1 and Table 2 shows. Moreover, since only the outermost envelope is important for the pulsations (see below), it is not surprising that the polytrope $n = 2$ (Prasad and Gurm 1961) does not give as good agreement as the standard model.

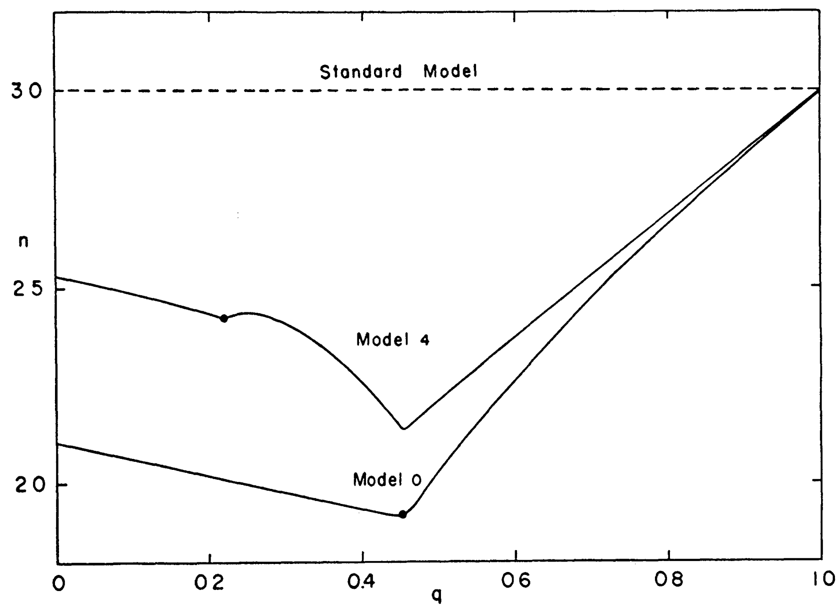

FIG. 1.—Effective polytropic index as a function of mass fraction for the standard model and for models 0 and 4 of 15 $M_\odot$. A dot marks the boundary of the convective core.

As Table 2 of this paper and Table 1 of Schwarzschild and Härm (1959) show, ω_0^2 increases as the evolution proceeds. This happens despite the decrease of β and hence of $3\Gamma_1 - 4$ (cf. eq. [11]). Thus the increasing central condensation plays the dominant role. For instance, between models 0 and 4 for 15 $M_\odot$, the surface value of $3\Gamma_1 - 4$ changes by a factor 1.17, whereas $\rho_c/\bar{\rho}$ changes by 12.9. Of course, $\rho_c/\bar{\rho}$ is not strictly proportional to the ratio of integrals in equation (11), but it gives an idea of the much greater effect of the increasing central condensation on ω_0^2.

Its effect, however, is somewhat limited. For instance, between models 1 and 2 for 15 $M_\odot$, $\rho_c/\bar{\rho}$ doubles. Likewise, between models 3 and 4 it doubles again. However, the change of ω^2 in the latter case is much smaller than in the former. The reason is that ω^2 depends on the rate of decrease inward of ξ, which drops rapidly in the outer envelope, and therefore is insensitive to the precise interior conditions. Figure 2 shows this in the case of the five calculated modes for model 4 of 15 $M_\odot$. Since ξ is determined chiefly by the structure of the envelope and hence by the total luminosity, so is ω^2. Now we note that the luminosity increases much less between models 3 and 5 than between models 1 and 2.

In fact, since ω_0^2 is completely independent of the nuclear-energy generation and hence of the stellar radius, it is governed *only* by the total luminosity and the age (which fixes

β_c). With reference to equation (11), the luminosity gives us directly β_0 (from the last of eqs. [17] in Paper I), and so Γ_1, and the age determines the chemical inhomogeneity, and so the degree of central condensation (note the similarity of $\rho_c/\bar{\rho}$ at a given value of X_c for the various masses in Table 2). Quantities which are dependent on ω_0^2 as well as independent of R, such as Q_0 and the period ratios, are also almost the same from mass to mass, for a given value of ω_0^2.

d) *Discussion of Previous Work on Periods*

Comment can now be made regarding the apparent previous failure of radial pulsations to account for the observed periods. From the results of Table 3, we see that Ledoux and Walraven (1958) evidently used values for the stellar radius that were too large in obtaining the low "observed" value of $Q_0 = 0.027$.

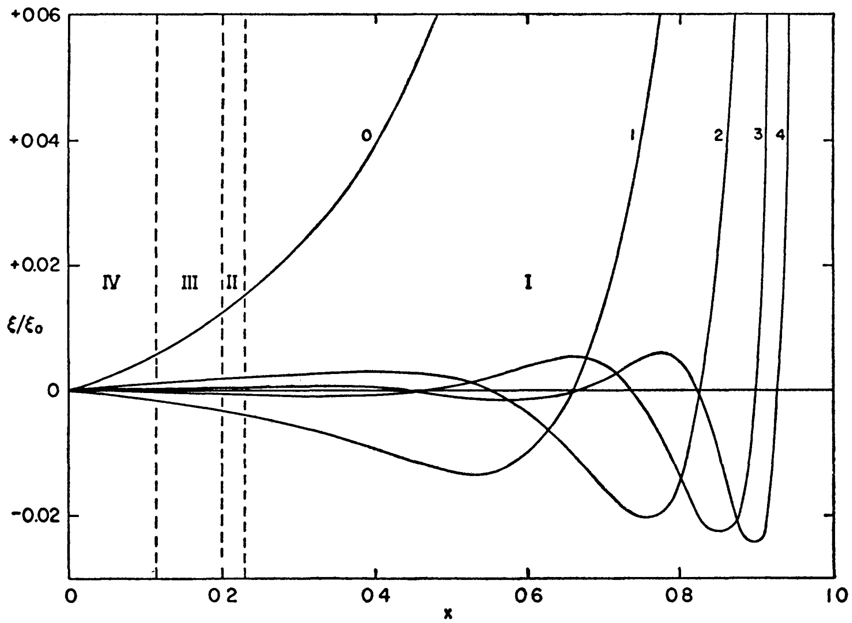

FIG 2.—Pulsation amplitude (normalized to value at the surface) as a function of radius fraction for model 4 of 15 $M_\odot$. Solutions are labeled with the mode number. Roman numerals designate the stellar zones (Paper I), which are marked off by vertical lines.

Gurm (1963) calculated the pulsational eigenvalues for Kushwaha's (1957) initial main-sequence model of 10 $M_\odot$ and $X_e = 0.90$, and found $\Pi_0 = 2.12$ hours. Since we see from Table 2 that the period more than doubles along the evolutionary track of massive stars, his conclusion that our present theory of stars on the upper main sequence seems inadequate because of the period disagreement appears to be unwarranted. Moreover, as we shall see in Section VI*a*, the periods (but not their ratios) may be considerably altered through small changes in the chemical composition.

Reddish and Sweet (1960) interpreted Struve's tentative suggestion of a secular period change in β Cephei in terms of the expanding radius during hydrogen burning. Their rough result that the rate of increase is an order of magnitude smaller than that required by observations is confirmed by our detailed models. However, in van Hoof's scheme slow period changes in β Cephei stars are to be explained by interacting modes.

V. DATA ON THE β CEPHEI STARS

In Table 4 the results of van Hoof's analyses of the light-curves of five β Cephei stars are presented. A similar analysis of ξ_1 Canis Majoris is not presented, since it was es-

Table 4

Observed Period Ratios in β Cephei Stars

β CMa		θ Oph	ν Eri	β Cep	β Cru	|
.03	Π_o (hours)	3.37	4.17	4.57	5.67	6
.679	Π_1/Π_o	0.659	0.675	0.674	0.678	0
.509	Π_2/Π_o	0.494	0.505	0.506	0.509	0
.407	Π_3/Π_o	0.395:	0.404	0.404	0.408:	0
.337	Π_4/Π_o	0.330:	0.336	0.336	0.340:	0
.750	Π_2/Π_1	0.750	0.748	0.751	0.751	0
.599	Π_3/Π_1	0.599:	0.599	0.599	0.602:	0
.496	Π_4/Π_1	0.501:	0.498	0.499	0.501:	0
962*d*	van Hoof	1962*b*	1961	1962*c*	1962*a*	1'

sentially an interpolation (van Hoof 1963). However, these stars are representative of all β Cephei stars, in that they include a broad range of periods, rotational velocities, and velocity amplitudes. The data of Table 4 were used to interpolate the models presented in Table 3, where only the adopted Π_1/Π_0 for each mass was listed, because the calculated and observed ratios of the higher modes agreed almost exactly.

To plot the observational data for β Cephei stars on the theoretical H-R diagram, we

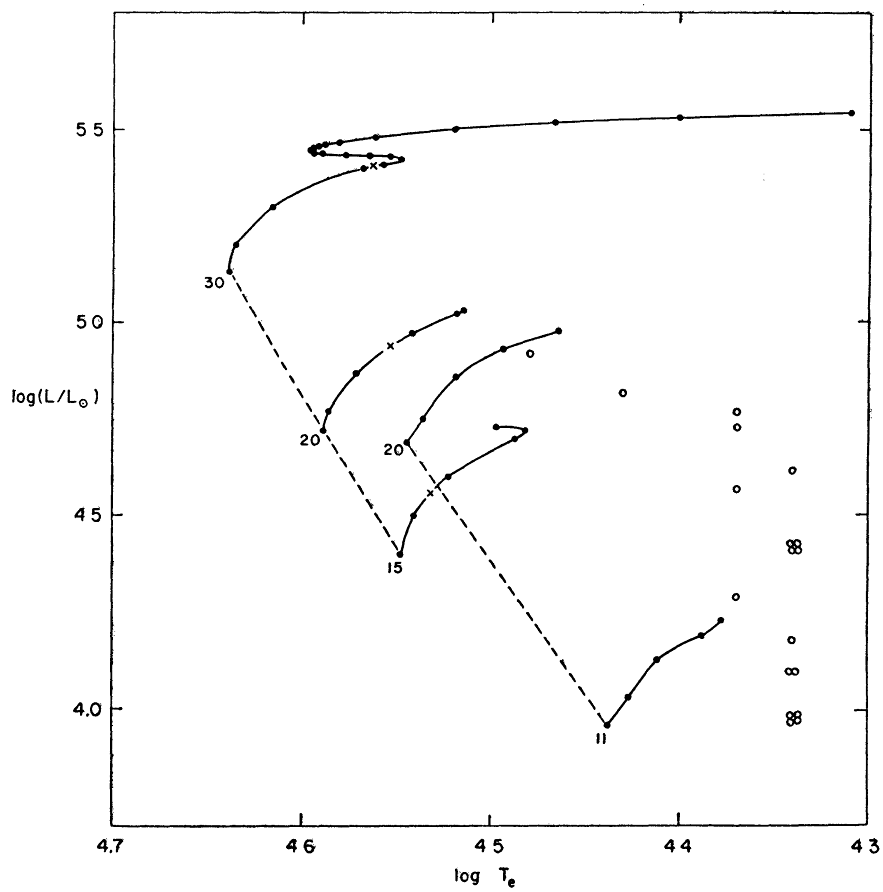

FIG. 3.—Theoretical H-R diagram of the upper main sequence, including observational points for the β Cephei stars. Model sequences for 15–20–30 $M_\odot$ are due to Stothers, and those for 11–20 $M_\odot$ to Henyey, LeLevier, and Levée. Crosses mark the interpolated models for equal calculated and observed period ratios.

have used the list given by van Hoof (1962*e*) and the relations between spectrum, effective temperature, and bolometric correction given by Harris (1963). Harris's bolometric corrections were used for all luminosity classes (II–IV). The luminosities were normalized by using $M_{\text{bol}} = +4.72$ for the Sun (Allen 1963). The resulting values of luminosity and effective temperature are not very different from those obtained by Schmalberger (1960).

The evolutionary tracks on Figure 3 came from Papers I, II, and the present paper for 15–20–30 $M_\odot$, and from Henyey *et al.* (1959) for 11–20 $M_\odot$. Crosses denote the interpolated models of Table 3.

It is clear from Figure 3, as well as Table 3, that the locus of constant-period ratios is not the locus of secondary contraction. Moreover, the constant-ratio strip falls to the left of the observed β Cephei strip, and the periods do not quite agree. However, the latter two discrepancies may be easily removed by a change in the radius. In the next section we shall see how this change may be accomplished, in the course of investigating the effect of changes in the relevant physical input quantities used in deriving the models.

VI. EFFECT OF CHANGING PHYSICAL PARAMETERS

a) *Chemical Composition*

In general, a change in the initial chemical composition of a star whose opacity is dominated by electron scattering will shift its evolutionary track along a band almost parallel to the initial main sequence in the H-R diagram. For, since $L \sim \mu_e^4$ and $R \sim \mu_e$, we have that $T_e \sim \mu_e^{1/2}$.

If we assume that all the β Cephei stars have the same mass and pulsate at the same evolutionary stage, but have different initial chemical compositions, then $\Pi \sim R^{3/2} \sim \mu_e^{3/2}$. Since Π is observed to vary by a factor of 2, we must conclude that X_e also varies by at least the same factor. This conclusion seems to be unjustified by observations of the upper main sequence. Moreover, if X_e varies, the period ratios will be different at the same evolutionary stage, in contradiction of the observed rough constancy of these ratios. Finally, the variation in X_e necessary to produce the observed variation in luminosity is much more than a factor of 2.

The last argument would also rule out the possibility of pulsation at differing evolutionary stages, even though supplementary calculations indicate that the constant-ratio strip on the H-R diagram for stars of the same mass but differing initial chemical composition falls in the same way as in Figure 3. The reason for the similar position of the strip is that ω_0^2 and the period ratios at the same evolutionary stage (l_c) are smaller for stars of lower X_e (and hence higher luminosity).

If, however, we assume with Schmalberger (1960) that the β Cephei stars have different masses but that all fall along the locus of secondary contraction ($X_c \approx 0.03$) on the H-R diagram, we can compute the necessary initial chemical composition for each mass on the basis of constancy of the period ratios. From Table 3 the interpolated model for 30 $M_\odot$ with $X_e = 0.70$ lies close to this locus. Using it as a standard, we invoke homology arguments to calculate X_e for other masses. From Paper I, as long as β_c is not too low, the dimensionless structure of the star is specified only by the parameters A and C at a given evolutionary stage (l_c), since the variable j may be replaced by $l^{0.285}$ (Schwarzschild 1958). Then, holding A and C constant, we calculate from $A \sim \mu_e^4 M^2$ that X_e must be 0.46 and 0.32, for 20 $M_\odot$ and 15 $M_\odot$, respectively. Since the pulsational eigenvalues, and hence the period ratios, will not change (because they depend only on the dimensionless structure of the star), these values of X_e are required for the hypothesis of secondary contraction. They are unrealistically low, and must be lower still because the mass at which $X_e = 0.70$ should actually be greater than 30 $M_\odot$, and because the decreased values of X_e imply an insufficiently decreased luminosity, through the lower opacity and the effective relation $L \sim M$ instead of $L \sim M^3$. Therefore, to give agreement with the observed luminosities, even smaller masses and lower X_e would have to be taken. Moreover, there is no reason to believe that the calculated periods would agree with the observations. In any case, it is difficult to see why β Cephei stars of lower mass should have lower initial hydrogen abundances.

Ideally, if the β Cephei strip were sufficiently well defined observationally, the stellar masses and chemical compositions could be determined with greater accuracy. For a given model, specified by M and μ_e, the evolutionary track crosses the strip on the H-R diagram at a certain point, where both Π_0 and the period ratios must agree with the observations. Since Π_0 depends essentially on the stellar radius and the period ratios on the luminosity, the required model is therefore uniquely determined.

b) Nuclear-Energy Generation

The radius of massive stars is essentially fixed by the rate of nuclear-energy generation. Since $\Pi \sim R^{3/2}$, it is clear that we may seek agreement with the observed periods by adjusting ϵ_0 or X_{CNO}, as well as by changing μ_e. To change Π_0 of the model for 15 $M_\odot$ that best fits the observations of period ratios, from 3.7 hours to the observed value of 5 hours, we require an increase in ϵ_0 or X_{CNO} by a factor 45. Such an increase seems inadmissible. Moreover, adjustment to agree with observations at 15 $M_\odot$ produces too great a Π_0 at 20 $M_\odot$.

c) Opacity

The inability of a reasonable change in the nuclear-energy generation rate to produce agreement with the observed periods will not be fatal, however. In our models we neglected opacity sources other than electron scattering, and it is certain that bound-free absorption processes will contribute non-negligibly in the outer envelope. The model sequence for 20 $M_\odot$ with $X_e = 0.68$, computed by Henyey *et al.* (1959), included these processes, and the resulting evolutionary sequence lies on the H-R diagram parallel to our sequence at very nearly the same luminosity (cf. Fig. 3). It is, however, displaced to lower effective temperatures by an amount equivalent to a change in log $(R/R_\odot)$ of 0.06, after allowance for differences in X_e, X_{CNO}, and ϵ_0. This change is brought about almost directly by the opacity, since we have that $R \sim \kappa$ from dimensional analysis of the equation of radiative energy transport. The change produces an increase of Π_0 from 5.2 to 6.4 hours, in the *direction* of agreement with observations.

The question arises whether inclusion of bound-free absorption will change the pulsational eigenvalues. Undoubtedly it will to some extent, but the ratios of the modes, especially those of the higher modes, should remain fairly constant, because they are nearly independent of the radius. We recall that it is these ratios that essentially fix the β Cephei strip on the evolutionary H-R diagram.

It may easily be shown, however, that the inclusion of bound-free absorption, which has an increasing effect at lower masses, actually serves to offset the line of constant period ratios *farther* from the locus of secondary contraction. Its inclusion is roughly equivalent to increasing X in the electron-scattering opacity. As discussed in Section VI*a*, the period ratios will then also increase. Therefore, at lower masses, the model for which the ratios agree with observations lies closer to the initial main sequence.

d) Semiconvection

Since the extent of semiconvection in a star depends mainly on the luminosity, a lowering of the initial hydrogen abundance in stars of a given mass increases the amount of semiconvection, through the relation $L \sim \mu_e^4$. However, supplementary calculations show that the semiconvective zone, as we have treated it, is unable to alter the pulsational eigenvalues to a perceptible degree, even in the last hydrogen-burning model of a star of 30 $M_\odot$. Hence uncertainties in the semiconvective theory will probably not be reflected in the pulsational characteristics of massive stars. For determination of these characteristics, it is adequate merely to consider the intermediate zones as wholly radiative.

e) Mass Loss

If the theory of Struve and Odgers (Struve 1955*b*) is correct, the β Cephei phenomenon may be explained by the ejection, deceleration, and subsequent infall of an atmosphere. In any case, it is to be expected that some mass will be lost (Sahade in Reddish and Sweet 1960). We should like now to examine whether the β Cephei (constant-ratio) strip is actually the evolutionary track of a star losing mass.

If the star remains chemically inhomogeneous, a constant-ratio line cannot be maintained since X_c along the line must *increase* as the stellar mass is lower (Table 3). If, however, the incipient instability causes and then maintains complete mixing of the stellar material, a constant-ratio line might be maintained since X_c $(=X_e)$ decreases with the mass. For average values of M and L taken from Table 1, the lifetime for a star of initially 20 $M_\odot$ to reach 10 $M_\odot$ will be $\Delta\tau = E\,\Delta X\,\bar{M}/\bar{L} = 10^7$ years. Hence the mean rate of mass loss is 10^{-6} $M_\odot$/year; this rate might not be unreasonable.

Three arguments seem to rule out complete mixing, however. First, the mass would be forced to decrease with X_e as μ_e^{-2} in order to preserve constancy of the period ratios (see Sec. VI*a*). Second, since $L \sim \mu_e^4 M^3/(1 + X_e) \sim M/(1 + X_e)$, and both M and $1 + X_e$ decrease by about the same factor, L will not change very much, in contradiction to the observations. Third, complete mixing not only restores stars to the initial main sequence, but as hydrogen is consumed, it produces a track to the *left*. A compromise based on partial mixing may be ruled out by the same argument applied against the inhomogeneous case.

We conclude that little mass loss occurs during the β Cephei phase, and in the absence of any direct observational evidence to the contrary, we have assumed that stars must evolve *across* the instability strip. Since the strip is so narrow, the time scale of evolution across it must be small, and therefore the mass loss in any case will be small.

VII. INTERPRETATION OF THE H-R DIAGRAM

We should now like to see whether the rapid drop in luminosity along the constant-ratio line is more compatible with the *observations* than the gentler drop occurring strictly along the locus of secondary contraction. First, since no known β Cephei stars are members of a binary system, we have relied on the model calculations to place the mass limits at 10 and 20 $M_\odot$, roughly. Then our constant-ratio models predict a period-luminosity law $\Pi \sim L^{0.40}$ in this range. (Extrapolating from 15 $M_\odot$ to 10 $M_\odot$, we should actually have an exponent slightly less than 0.40.) Since Π will change by a roughly constant multiple for all masses if the constants determining R are changed, the exponent 0.40 will remain the same for horizontal shifts of the evolutionary tracks in the H-R diagram. Now all the β Cephei stars taken together (van Hoof 1962*e*) yield a law $\Pi \sim L^{0.25}$. However, only four of them have accurately determined luminosities. These are members of the Scorpio-Centaurus cluster, and include θ Ophiuchi and β Crucis from Table 4. They yield the law $\Pi \sim L^{0.35}$. Good agreement is therefore found with the theoretical law.

Second, the luminosity class drops from II–III for the variables of earliest spectral type (B0.5) to IV for those of latest spectral type (B2). This suggests that the β Cephei strip does indeed approach the main sequence closer than does the locus of secondary contraction, which should probably not show a drop, or at least a large one, in luminosity class. We note further that the magnitude difference between stars of classes IV and V attains a minimum at B2 (Arp 1958). From Figure 3 the constant-ratio strip, extrapolated, would run close to the initial main sequence at 10 $M_\odot$ (B2).

Third, observations of early-type clusters and associations indicate that the tip of the Trumpler turn-off, which is believed on evolutionary grounds to represent the point of secondary contraction, occurs at luminosity class III. For example, in I Gemini the turn-off from the initial main sequence appears at B1 V and the tip of the turn-off at B1 III (Crawford, Limber, Mendoza, Schulte, Steinman, and Swihart 1955). This suggests that B1 IV β Cephei stars would not have reached the end of hydrogen burning.

Fourth and finally, the near constancy of the effective temperature for all the models along the constant-ratio line (Table 3) is compatible with the observation (for cooler stars at least) that brighter luminosity classes are associated with cooler effective temperatures than fainter classes at the same spectral type (Arp 1958). Thus a B0 II star may have roughly the same temperature as a B2 IV star. In any case, the opacity argument of Section VI*c* points to corrected models with effective temperatures that would be somewhat lower for the stars of lower mass.

The observed spectral (or mass) range of the β Cephei stars is remarkably well defined. One indication that the *lower* mass cutoff should occur near B2 is that our calculations show an intersection of the extrapolated constant-ratio strip with the main sequence near 10 $M_\odot$. Second, noting that the β Cephei stars occur only among sharp-line (slowly rotating) early B stars, McNamara and Hansen (1961) ascribe the cutoff to increasing rotational velocities among the late-type B stars. (This increase is observed in both luminosity classes V and III [Allen 1963].)

An unfruitful suggestion regarding the *upper* limit to the mass is that semiconvection starts to become important in stellar envelopes at about 20 $M_\odot$. Although convection tends to damp pulsations, the semiconvective zone is too ineffective and lies too deep for this purpose (see Fig. 2).

Observational evidence exists, however, for the continuation of instability up to the highest masses. But in the case of the O giants, the instability manifests itself in the Wolf-Rayet phenomenon. Westerlund (1961) and Westerlund and Smith (1964) have shown that, in the H-R diagrams of O associations in the Large Magellanic Cloud, the single Wolf-Rayet stars always appear at the tip of the Trumpler turn-off. These authors suggest that the masses of the Wolf-Rayet stars lie between 20 and 60 $M_\odot$, which is what we require to explain them as an "extension" of the β Cephei strip, now occurring close to the locus of secondary contraction.

Sahade (1962) gives a table of computed masses for some of the galactic Wolf-Rayet stars. Although they appear to be less massive than their OB companions, the three luminosity classes given for the companions are all Class I. Hence we expect these companions to be more massive, since they presumably evolved further (past the Wolf-Rayet phase of pseudo-class O III). At any rate, a substantial rate of mass loss in Wolf-Rayet stars is not at all prohibited by the observations (Underhill, private communication).

Finally, in an analysis of Wolf-Rayet spectra Smith (1955) has reported occasional variability of emission-line intensities on a time scale of a few hours, in analogy with the Of stars (Oke 1954). We emphasize, however, that the observable form of the instability and probably the energizing mechanisms sustaining it are wholly different for the Wolf-Rayet and β Cephei stars. Why an apparent changeover should occur at B0 is unknown.

VIII. CONCLUSIONS

By evolving model sequences for stars of 15, 20, and 30 $M_\odot$ from the initial main sequence to the end of hydrogen burning, it has been possible to study the pulsational eigenfrequencies of massive stars in detail. Pulsationally, the stellar envelopes behave like polytropes of index $n = 3$, with some characteristic mean Γ_1. To some extent the masses form a homologous sequence, along lines of constant X_c for the stable models and constant ω_0^2 for the perturbed models. The increasing central condensation during evolution has a dominant effect on ω_0^2, although the luminosity (through Γ_1) somewhat modifies this result.

Comparison of the theoretical results on period ratios has been made with the observations of β Cephei stars. If van Hoof's period interpretation is correct, we conclude that these variables must be in the hydrogen-burning phase, even apart from the corroborating evidence of stellar statistics. Further, they must exhibit a range of masses from about 10 to 20 $M_\odot$ and have, presumably, roughly the same "normal" Population I chemical abundances. A wide variation in X_e, at least, is prohibited. The apparent discrepancy of the effective temperatures and periods with the observations arises from the inaccuracy of the calculated radii, and is due simply to our exclusion of opacity sources other than electron scattering. Rough allowance for this omission produces the needed agreement but does not otherwise alter our results. The extent and effect of semiconvection, mixing, and mass loss should be small. When mixing or mass loss has occurred extensively, stars are expected not to exhibit β Cephei behavior.

The main conclusion of the paper is that, on the assumption of the validity of van Hoof's work, β Cephei stars of lower mass will lie closer to the initial main sequence, whereas those of higher mass (and the Wolf-Rayet stars) should be almost at the end of hydrogen burning. Various pieces of observational evidence, comprising the period-luminosity law, luminosity classes, comparison with cluster H-R diagrams, and rough constancy of the effective temperatures, tend to support this expectation.

Thus it appears that radial pulsations of hydrogen-burning giants may explain at least some of the observations of β Cephei stars. However, the effect of rotation and possible mass loss cannot be definitely ascertained at this time. Further, theory cannot yet say why the β Cephei stars lie in the distinct range B0.5–B2 III–IV, nor why only some stars in this range become variable, nor what the sources maintaining several simultaneously excited modes may be.

A preliminary theoretical discussion of the β Cephei stars appeared in the author's doctoral dissertation, Harvard University (1963), which was supported in part by a Harvard scholarship during the first half of the academic year 1963–1964. Another part of the work reported in this paper was supported by an NAS-NRC postdoctoral resident research associateship under the National Aeronautics and Space Administration. It is a pleasure to thank Dr. Leon Lucy and a referee for helpful criticisms, and Dr. Robert Jastrow for his hospitality at the Institute for Space Studies.

REFERENCES

Allen, C. W. 1963, *Astrophysical Quantities* (London: Athlone Press).
Arp, H. 1958, *Hdb. d. Phys.*, ed. S. Flügge (Berlin: Springer-Verlag), **51**, 75.
Böhm-Vitense, E. 1963, *Pub. A.S.P.*, **75**, 154.
Chandrasekhar, S. 1939, *An Introduction to the Study of Stellar Structure* (Chicago: University of Chicago Press).
Chandrasekhar, S., and Lebovitz, N. R. 1962, *Ap. J.*, **136**, 1105.
Crawford, D., Limber, D. N., Mendoza, E., Schulte, D., Steinman, H., and Swihart, T. 1955, *Ap. J.*, **121**, 24.
Ezer, D., and Cameron, A. G. W. 1963, *Icarus*, **1**, 422.
Gurm, H. S. 1963, *M.N.*, **126**, 419.
Harris, D. L. 1963, *Basic Astronomical Data*, ed. K. Aa. Strand (Chicago: University of Chicago Press), p. 263.
Hayashi, C., and Cameron, R. C. 1962, *Ap. J.*, **136**, 166.
Henyey, L. G., LeLevier, R., and Levée, R. D. 1959, *Ap. J.*, **129**, 2.
Hoof, A. van. 1961, *Zs. f. Ap.*, **53**, 106, 124.
———. 1962*a*, *ibid.*, **54**, 244.
———. 1962*b*, *ibid.*, p. 255.
———. 1962*c*, *ibid.*, **56**, 15.
———. 1962*d*, *ibid.*, p. 27.
———. 1962*e*, *Kleine Veröff. d. Remeis-Sternwarte Bamberg*, No. 34, p. 68.
———. 1963, *Zs. f. Ap.*, **56**, 141.
Kopylov, I. M. 1959, *Izv. Krymsk. Astrofiz. Obs.*, **21**, 40.
Kushwaha, R. S. 1957, *Ap. J.*, **125**, 242.
Ledoux, P. 1939, *Ap. Norvegica*, **3**, 193.
———. 1951, *Ap. J.*, **114**, 373.
———. 1958, *ibid.*, **128**, 336.
Ledoux, P., and Pekeris, C. L. 1941, *Ap. J.*, **94**, 124.
Ledoux, P., and Walraven, Th. 1958, *Hdb. d. Phys.*, ed. S. Flügge (Berlin: Springer-Verlag), **51**, 353.
McNamara, D. H., and Hansen, K. 1961, *Ap. J.*, **134**, 207.
Oke, J. B. 1954, *Ap. J.*, **120**, 22.
Oliinik, G. T., and Porfir'ev, V. V. 1963, *Astr. Zhur.*, **40**, 774.
Prasad, C., and Gurm, H. S. 1961, *M.N.*, **122**, 409.
Reddish, V. C., and Sweet, P. A. 1960, *Modèles d'étoiles et évolution stellaire* (extrait des *Mém. Soc. R. Sci. Liège*, 5th ser., Vol. **3**), p. 263.

Sahade, J., ed. 1962, *Symposium on Stellar Evolution* (La Plata: National University of La Plata), p 185.
Sakashita, S., Ôno, Y., and Hayashi, C. 1959, *Prog Theoret. Phys* (Kyoto), **21**, 315.
Schmalberger, D. C. 1960, *Ap J.*, **132**, 591
Schwarzschild, M. 1941, *Ap. J.*, **94**, 245
———. 1958, *Structure and Evolution of the Stars* (Princeton, N J : Princeton University Press)
Schwarzschild, M., and Härm, R. 1959, *Ap J.*, **129**, 637.
Smith, H. J. 1955, unpublished dissertation, Harvard University.
Stothers, R. 1963, *Ap. J.*, **138**, 1074 ("Paper I")
———. 1964, *ibid.*, **140**, 510 ("Paper II")
Stothers, R., and Schwarzschild, M. 1961, *Ap. J* , **133**, 343.
Struve, O. 1955*a*, *Pub. A.S.P.*, **67**, 29
———. 1955*b*, *ibid.*, p. 135.
Westerlund, B. 1961, *Uppsala Ann* , Vol **5**, No. 1
Westerlund, B E , and Smith, L F. 1964, *M N* , **128**, 311.